14/17

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A contraction approach to periodic optimization problems

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Centre for Applied Research at NHH
Bergen, November 2017
ISSN 1503-2140

Contents

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November 6, 2017

Abstract

Consider an infinite horizon, multi-dimensional optimization problem with arbitrary but finite periodicity in discrete time. The problem can be posed as a set of coupled equations. We show that the problem is a special case of a more general class of problems, that the general class has a unique solution, and that the solution can be obtained with the help of a contraction operator. Special cases include the classical Bellman problem and stochastic problem formulations. Thus, we view our approach as an extension of the Bellman problem to the special case of non-autonomy that periodicity represents, and we thereby pave the way for consistent and rigorous treatment of, for example, seasonality in discrete, dynamic optimization. We demonstrate our method in a simple example with periodic variation in the objective function.

Key words: Bellman, optimization, periodicity, contraction operator, solution scheme.

1. Introduction

Periodicity is an important characteristic of many systems that are subject to control. Rigorous treatment of periodicity in optimization problems is non-trivial because periodicity is a special case of non-autonomy. Non-autonomy typically renders many optimal control problems difficult and costly to deal with or even intractable. Thus, periodicity in applications is often treated in some ad-hoc manner or abstracted from altogether, for example by considering the aggregate or mean forcing. To our knowledge, periodicity in infinite horizon optimal control problems in discrete time is not treated formally in the theoretical literature. It turns out that the periodic problem is a special case of a general class of problems that can be shown to be fix-point problems for a contraction operator. The contraction operator can be used to obtain the solution in an iterative procedure. The class of problems that we study include the classical Bellman problem, the periodic problem formulations of original interest, stochastic problems, and further, more esoteric formulations. Our key contribution is nevertheless an extension of the classical Bellman result to the special case of non-autonomy that periodicity represents.

Examples of periodicity in decision problems include demand systems subject to supply control. In particular, annual or seasonal, even weekly, cycles in demand is well-known for electricity (Cappers *et al.* 2010) and energy in general, and a broad range of consumer goods have seasonal fluctuations in demand; see McClain and Thomas (1977) for an early linear programming approach to seasonal demand, or Bradley and Arntzen (1999) for a mixed-integer algorithm. Nagaraja *et al.* (2015) provide a brief and recent review of related theory on seasonal demand problems. Other dynamic decision problems with periodic features are found in transport and logistics systems subject to routing control (see Liebchen 2008 on the use of optimization in the periodic event-scheduling problem), or natural systems subject to management control. For example, renewable resources such as fish stocks may have periodicity in growth or other natural processes as well as periodicity in prices and costs; see Smith (2012) and Huang and Smith (2014).

To illustrate our approach to periodic problems, we apply our numerical scheme to a stylized decision problem with periodicity in the objective function. The example serves to demonstrate the feasibility of our approach, and also suggests significant, practical implications of taking periodicity explicitly into account. In particular, the solution of the periodic problem has features that are not typically present in problems with no periodicity.

Given the prevalence of periodic characteristic of many systems subject to control, we think our contribution is important and highly valuable. We show that the classical Bellman problem approach can be extended to periodic problems, and that this extension is, while nontrivial, both conceptually and numerically feasible and practical. Ultimately, a broader class of problems can be treated with our approach, but the Bellman problem and the periodic problem are directly applicable to real-world decision problems and thus we keep our focus on these formulations. Further, as the periodic problem is the motivation for considering the problems we target in our most general result, we start out our analysis by showing how the general problem formulation suggests itself from the periodic problem setup.

2. A contraction operator for the periodic problem

A general, infinite horizon, autonomous, dynamic, discounted, discrete-time optimization problem considers the following:

$$\max_{\{u_k\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \beta^{k+1} \cdot \Pi(x_k, u_k) \quad (1)$$

such that $x_{k+1} = F(x_k, u_k)$, $u_k \in U(x_k)$, $k = 0, 1, 2, \ldots$, and $x_0 \in X$ given. $0 < \beta < 1$ is a discount factor. $X \subset \mathbb{R}^n$ is the feasible state space, x_k is the dynamic state variable at the beginning of time interval[1] k. $U: X \to \mathbb{R}^p$ is a nonempty and compact valued correspondence that specifies the admissible controls u_k at the state x_k. That is, u_k is the decision or control variable that has to be decided for each instant of the infinite time sequence $\{t_0, t_1, t_2, \ldots\}$. $\Pi: X \times U \to \mathbb{R}$ is bounded and continuous and gives the performance measure (return) *at the end* of each interval. $F: X \times U \to X$ is a continuous operator that governs the state variable such that $x_{k+1} = y_k$ is the state at the beginning of interval $k + 1$. With these conditions in place, optimal controls $\{u_k^*\}_{k=0}^{\infty}$ and corresponding paths $\{x_k^*\}_{k=0}^{\infty}$ exist, as does the value function of the problem, $V(x) = \sum_{k=0}^{\infty} \beta^{k+1} \cdot \Pi(x_k^*, u_k^*)$ with $x_0 = x$. The value function is the fixed point of the Bellman operator T_B, which is defined on the space $BC(X)$ of real, bounded, and continuous functions on X and given by

$$T_B V = \max_{u \in U(x)} \{\beta \cdot \Pi(x, u) + \beta \cdot V(y)\} \quad (2)$$

with $V \in BC(X)$ and $y = F(x, u)$. See Bertsekas (2001) for a more general treatment of problems of type (1).

We now consider the non-autonomous but periodic problem where $\Pi_k(x, u)$ is the return function and $F_k(x, u)$ is the time evolution operator for interval k. Sets for the feasible states ($X_k \subseteq X$) and admissible controls (U_k) may also vary between intervals. The control set may vary with the state such that we have $U_k = U_k(x_k)$, but we typically omit the state argument.

[1] We use the term ‘interval’ rather than ‘period’ here and reserve the latter to denominate the periodic length characteristic.

The problem is periodic in the sense that for a finite integer $N \geq 1$, we have $\Pi_k = \Pi_{k+N}$, $\mathrm{F}_k = \mathrm{F}_{k+N}$, $X_k = X_{k+N}$, and $U_k = U_{k+N}$. We say that the problem is periodic with period N and that the performance or return measure and the dynamic constraint functionally repeats themselves. Without adding complexity, we can allow for varying interval length. Thus, each different interval has potentially different discount factor values. We write the length of interval k as $T_k = t_k - t_{k-1}$ and its discount factor as β_k. Periodicity implies $T_k = T_{k+N}$ and $\beta_k = \beta_{k+N}$. The length of the cycle of N intervals is then $T = \sum_{i=1}^{N} T_i = t_N - t_0$, and the discount factor for the cycle of N intervals is $\beta = \prod_{i=1}^{N} \beta_i$. Figure 1 accounts for interval index references.

Figure 1: Interval index reference for periodic problems. Note that the return (Π_k) is yielded at the end of interval k, but that V_k refers to the beginning of interval k.

Although a real discounted problem cannot have periodic present value, the running value will be periodic under a constant per time discounting if involved operators (Π_k, F_k) or spaces (U_k, X_k) are periodic. That is, for a problem of type (1) to be periodic, one or more of Π_k, F_k, U_k, and X_k need to have a periodic feature. As suggested above, a periodic feature is such that it repeats itself with some inherent period. If a problem includes several periodic features, the problem period N has to be the least common multiple of the potentially different inherent periods of the different features.

The Bellman equation for the problem in (1) is, using the operator defined in (2), written simply as $V = T_B V$. The periodic problem intuitively suggests the set of N nested equations:

$$\begin{aligned} V_k(x) &= \max_{u_k \in U_k(x)} \{\beta_k \Pi_k(x, u_k) + \beta_k V_{k+1}(x')\}, \quad k = 1, \dots, N-1 \\ V_N(x) &= \max_{u_N \in U_N(x)} \{\beta_N \Pi_N(x, u_N) + \beta_N V_1(x')\} \end{aligned} \tag{3}$$

In (3), $x' = F_k(x, u_k)$ is shorthand notation for the state variable one interval ahead. If V_k is interpreted as the value function for interval k, the equation set (3) follows from value additivity with its inherent economic logic that present value is what you earn presently plus the discounted value of future earnings. ‘Earn’ is not necessarily meant in its strict, monetary sense, but can be any type of utility-like flow.

Now, consider functional equations of type

$$\mathcal{V}(x) = \mathcal{T}\mathcal{V}(x) \tag{4}$$

where $\mathcal{V}(x)$ is an N-dimensional bounded vector function in $B(X)$, and further $x \in X \subset \mathbb{R}^n$. The components of the operator $\mathcal{T}$ are defined as

$$\mathcal{T}_k\mathcal{V}(x) \triangleq \max_{u \in \Gamma_k(x)} \{ \widehat{\Pi}_k(x,u) + \beta_k \mathcal{L}_k \mathcal{V}(x') \}, \quad k \in (1, \ldots N) \tag{5}$$

In (5), $\mathcal{L}_k$ are Lipschitz operators with Lipschitz constants γ_k, $\widehat{\Pi}_k(x,u)$ are bounded functions, and the correspondence $\Gamma_k(x)$ specifies admissible sets. $\widehat{\Pi}_k(\cdot)$ may take on two forms depending on the timing of the return. If returns are realized at the end of each interval, as in (3), the returns are discounted and we have $\widehat{\Pi}_k(x,u) = \beta_k \Pi_k(x,u)$. If, on the other hand, returns are realized at the beginning of each interval, returns are not discounted and we have $\widehat{\Pi}_k(x,u) = \Pi_k(x,u)$. The parameters $\beta_k \in (0,1)$. As we will argue below, (3) is a special case of (4) with $\mathcal{T}$ defined by (5).

The definition in (4) is further a special case of two different stochastic formulations. Let $z \in Z \subseteq \mathbb{R}^q$ be a real-valued, q-dimensional vector of stochastic elements that are realizations of a known, stochastic process (that is, the probability transition function $Q(z, d\mu_z)$ is known and the expectation operator E_z is well-defined, see Stokey *et al.*, 1989, p. 241). The stochastic elements can be present in both the return functions and the operators governing the state variables that are considered to be Markov decision processes. We thus write $\Pi_k(x,u,z)$ and $F_k(x,u,z)$; these are both measureable. If both present and future realizations of the stochastic process is uncertain, we consider the following definition of $\mathcal{T}$:

$$\mathcal{T}_k\mathcal{V}(x) \triangleq \max_{u \in \Gamma_k(x)} E_z\{ \widehat{\Pi}_k(x,u,z) + \beta_k \mathcal{L}_k \mathcal{V}(x') \}, \quad k \in (1, \ldots N) \tag{6}$$

In (6), $E_z\{\cdot\}$ is the expectancy operator with regard to z. The definition in (6) aligns with the typical formulation in Bertsekas (2001). Other problem formulations, however, consider the present realization of the stochastic process as known. Such formulations require the stochastic elements to be considered as part of the state vector. Consider $s = (x,z)$ as an extension of the state and consider the following definition of $\mathcal{T}$:

$$\mathcal{T}_k\mathcal{V}(s) \triangleq \max_{u \in \Gamma_k(s)} \{ \widehat{\Pi}_k(s,u) + \beta_k E_{z'} \mathcal{L}_k \mathcal{V}(s') \}, \quad k \in (1, \ldots N) \tag{7}$$

The definition (7) aligns with the typical formulation in Stokey *et al.* (1989). By inspection, we see that (5), the deterministic case, is a special case of both (6) and (7). The following theorem holds for all these potential definitions of $\mathcal{T}$; (5), (6), or (7).

Theorem: $\mathcal{T}$ is a contraction operator on bounded vector functions if $\eta \triangleq \max\{\beta_k\gamma_k | k = 1, \ldots, N\} < 1$.

Proof: Let $\mathcal{V}(x)$ and $\mathcal{W}(x)$ be arbitrary elements in $B(X)$ and let $|| \cdot ||$ denote the sup-norm. For component k, we have:

$$\begin{aligned} \mathcal{T}_k\mathcal{V} &= \mathcal{T}_k(\mathcal{W} + \mathcal{V} - \mathcal{W}) \\ &\leq \mathcal{T}_k\mathcal{W} + ||\beta_k E_{z'}\mathcal{L}_k(\mathcal{V} - \mathcal{W})|| \\ &\leq \mathcal{T}_k\mathcal{W} + \beta_k||\mathcal{L}_k(\mathcal{V} - \mathcal{W})|| \\ &\leq \mathcal{T}_k\mathcal{W} + \beta_k\gamma_k||\mathcal{V} - \mathcal{W}|| \end{aligned} \tag{8}$$

The first inequality in (6) follows from properties of the sup-norm. The second inequality follows from the expectancy operator having a Lipschitz constant of one. The final inequality follows from properties of the Lipschitz operator $\mathcal{L}_k$. From (8), we have $\mathcal{T}_k\mathcal{V} - \mathcal{T}_k\mathcal{W} \leq \beta_k\gamma_k||\mathcal{V} - \mathcal{W}||$. We can revert the roles of $\mathcal{V}$ and $\mathcal{W}$ in (8) to obtain $\mathcal{T}_k\mathcal{W} - \mathcal{T}_k\mathcal{V} \leq \beta_k\gamma_k||\mathcal{V} - \mathcal{W}||$ and conclude:

$$|\mathcal{T}_k\mathcal{V} - \mathcal{T}_k\mathcal{W}| \leq \beta_k\gamma_k||\mathcal{V} - \mathcal{W}|| \tag{9}$$

Inequality (9) holds for all k, and we have

$$||\mathcal{T}\mathcal{V} - \mathcal{T}\mathcal{W}|| \leq \eta||\mathcal{V} - \mathcal{W}|| \tag{10}$$

where $\eta \triangleq \max\{\beta_k\gamma_k | k = 1, \ldots, N\}$. That is, $\mathcal{T}$ is a contraction operator if $\eta < 1$, and this concludes the proof.

If $\mathcal{T}$ is operating on a compact function space, for example $B(X)$, then $\mathcal{T}$ has an existing and unique fix-point. Because $\mathcal{T}$ is a contraction, the fix-point can be obtained by iterations.

For our result to apply to the periodic problem, it remains to show that (3) is a special case of (4) and that the requirement on η holds. The left-hand sides of (3) and (4) are identical by definition. We thus need to show that the right-hand side in (3), for all k, is a special case of (5) which defines the right-hand side of (4). Because we have proved contraction also for the stochastic formulations in (6) and (7), our result also applies to stochastic analogous extensions of (3). We summarize this result in the following corollary:

Corollary: The periodic optimization problem in (3) and analogue stochastic problems are contraction problems and have unique solutions, that is, the value functions.

Proof: The operator defined by $\mathcal{L}_k V \triangleq V_i$ for all k, with $i = k + 1$ for $k \in (1, \ldots, N - 1)$ and $i = 1$ for $k = N$, is a Lipschitz operator with Lipschitz constant $\gamma_k = 1$. That is, (3) is a special case of (4). The β_k in (3) are discount factors, and for all k we have $\beta_k < 1$. Thus, $\eta < 1$, and the corollary follows from the theorem.

The proof of the corollary can be readily modified to show that also the classical Bellman problem (that is, set $i = k$ for all k in the proof) is a special case of (4), as is any choice for $i \in \{1, \ldots, N\}$. Further, there exists a huge set of Lipschitz operators that fulfill the requirements of the theorem, and potential applications of (4) are many.

Please note that when we use the sup-norm in the theorem above, it represents the worst-case scenario with regard to convergence. Thus, in most applications, we expect convergence to be faster than that implied by η.

Varying interval length requires suitable adaptions of Π_k, F_k, X, and Y_k (or the comparable stochastic elements), as well as the following specification of β_k. If interval k represents a share δ_k of the N-cycle, such that $t_k - t_{k-1} = \delta_k \cdot (t_N - t_0)$, we have $\beta_k = \beta^{\delta_k}$. In many applications, the N-cycle represents a year, and β is then the annual discount factor. The extension to varying interval length is an important and useful extension, not least because it allows for reductions in dimensionality. For example, consider a problem that is formulated on an annual level, but where one month is different such that the problem is periodic. Without the option of varying interval length, the model would have $N = 12$. With varying interval length, $N = 2$ suffices.

We have established a numerical routine based on the above results – using the set of equations in (3). Below, we apply this routine to an example which suggests that taking account of periodicity may have significant practical implications. The numerical results were obtained from code written in standard FORTRAN.

3. Applications

We derived the above results while working on periodic optimization problems. The major innovation is to consider a vector function rather than a scalar value function. The use of a vector function and our theorem above may be useful in applications other than periodic optimization problems. In the following, we discuss some potential applications and discuss how problems may be formulated for our result to apply. We presume here that (6) is a suitable definition, but depending on the problem formulation, (5) or (7) may be better suited.

An application closely related to periodic optimization problems is finite time optimization problems. Finite time optimization problems are typically solved by backward induction. Such solutions may be cumbersome to conciliate with given initial values. Our approach applies directly, however, where each interval is represented by an element in the vector function. Any form of non-autonomy may be accommodated (as with backward induction). Thus, for interval k, we have

$$V_k(x) \triangleq \max_{u \in \Gamma_k(x)} E_z\{\widehat{\Pi}_k(x,u,z) + \beta_k V_{k+1}(x')\}, \quad k \in (1, \dots N) \tag{11}$$

With $V_{N+1}(x) = G(x)$ representing salvage value, (11) can be interpreted as a finite time optimization problem with N periods. The above corollary applies and shows that (11) is a contraction problem that in general can be solved. Solutions are on general feedback form that are readily conciliated with any given initial value. The argument of the above corollary shows that the theorem applies and thus that (4), with (11) defining the vector function, has a unique and existing solution.

Some game theoretic problems may also be addressed by our methodology. Consider dynamic games over infinite time with non-cooperative (self-serving) behavior of N agents, but where the decisions of others partly or fully influence the return of any individual agent. Many common-pool resource games (Ostrom *et al.* 1994) fall within this type of games. For agent k, the problem is to maximize over one's own decisions while taking account of the decisions of others on both the current return and future returns. Further, decisions may depend or be restricted by a state vector x. Elements in the state vector may be common or private goods. The problem can be formulated as follows:

$$V_k(x) \triangleq \max_{u_k \in \Gamma_k(x)} E_z\{\widehat{\Pi}_k(x,u_k,u_{-k},z) + \beta_k V_k(x')\}, \quad k \in (1, \dots N) \tag{12}$$

The notation $\widehat{\Pi}_k(x,u_k,u_{-k},z)$ makes explicit that the return for agent k depends on the decisions of that agent (u_k) and the decisions of all other agents (u_{-k}). That the return function depends on the entire vector of decision variables ($[u_1, \dots, u_N]$) necessitates the consideration of a vector function ($[V_1, \dots, V_N]$). The proof of the corollary can be modified (with $i = k$) to show that (4) with (12) defining the vector function may have a unique feedback solution. It relies on the specificities of the game and whether or not they imply the properties needed for the various sets involved.

Multi-objective optimization, where multiple objective functions are dealt with, is pursued mainly along two methodological tracks (Deb 2005). One method is to assign preference weights to the different objective functions and consider the weighted sum of objectives with classical (single-objective) optimization methods. A second method considers the different objectives separately with classical methods to produce a frontier of solutions, for then to pose a second problem of choosing a final solution on this frontier. Both methods require 'higher-level' information, either to construct preference weights or to solve the frontier problem (Deb 2005). We suggest to consider the different objective functions separately, consistent with the first method just described, but simultaneously in vector form. For objective k, the problem is as follows:

$$V_k(x) \triangleq \max_{u\in\Gamma_k(x)} E_z\{\widehat{\Pi}_k(x,u,z) + \beta_k V_k(x')\}, \quad k \in (1,\dots N) \tag{13}$$

The objective $\widehat{\Pi}_k(\cdot)$ depends on parts of or the full state vector x and the decision vector u. Complicated interdependencies can thus be allowed, for example where two objectives share no state or decision variables and thus may be considered separately unless they both share state or decision variables with a third objective. The optimization problem implied by (13) is identical to that implied by (11), and, as argued above, our approach may apply.

4. A simple one-dimensional example

To illustrate the use of our method and the numerical scheme, we return to the problem that lead to the developments above. In addition to demonstrating our method, the following example shows the relevance and potential importance of considering periodic features in operational decision problems.

We consider a growth model typically used in natural capital management studies. The problem has periodic features with different interval lengths. We have

$$x_{k+1} = F_k(x_k) - u_k \tag{14}$$

where x_k is the level of natural capital at the begining of interval k, u_k is the level of exploitation, and the function F_k is a fourth-order Runge-Kutta discrete time representation of a typical continuous growth function; a modified logistic growth function (see Smith 2012 and references therein). We consider $y_k = x_{k+1}$ as our decision variable (u_k is eliminated with $u_k = F_k(x_k) - y_k$). The return functions are

$$\Pi_k(x_k, y_k) = F(x_k) - y_k - \delta_k^{-\gamma} c_k \cdot (F(x_k) - y_k)^{1+\gamma} \tag{15}$$

where c_k is an exploitation cost parameter, and $\gamma > 0$ is a convexity parameter.[2] Assume exploitation costs are periodic, with relative brief intervals of low cost and normal costs in the remaining intervals. The cost parameter is half of the normal level when low, and we write this as $c_{LC} = c_{NC}/2$; subscripts refer to *low cost* and *normal cost*. There are thus two intervals each period, $N = 2$, and $\delta_{LC} = 1/4$, $\delta_{NC} = 3/4$. That is, if a period is one year, the low cost interval is a quarter and the normal cost interval is three quarters. The difference in cost may reflect a

[2] Equation (15) derives from the typical expression $u_k - \delta_k c_k (u_k/\delta_k)^{1+\gamma}$ where the exploited capital has a unit price and costs are convex and depends on the exploitation rate; again see Smith (2012) and references therein.

situation where the resource, for example a fish stock, concentrates (seasonally) in a smaller area to spawn.

We solve (3), subject to (14) and (15), numerically and derive periodic optimal feedback decision rules as functions of the capital level at the beginning of each interval: $y_{LC}(x)$ and $y_{NC}(x)$. Figure 2 reports these decision rules together with the replacement curve (the 45-degree line). If the curves of the decision rules are below the replacement curve, the capital level is effectively reduced in the given interval and for the given initial capital level. The replacement curve also serves as the identity map used to transfer between subsequent periods ($y_k = x_{k+1}$). Figure 2 shows a dynamic path from the initial capital level $x_0 = 1{,}500$, which quickly converges to stable two-cycle at relatively high capital levels.

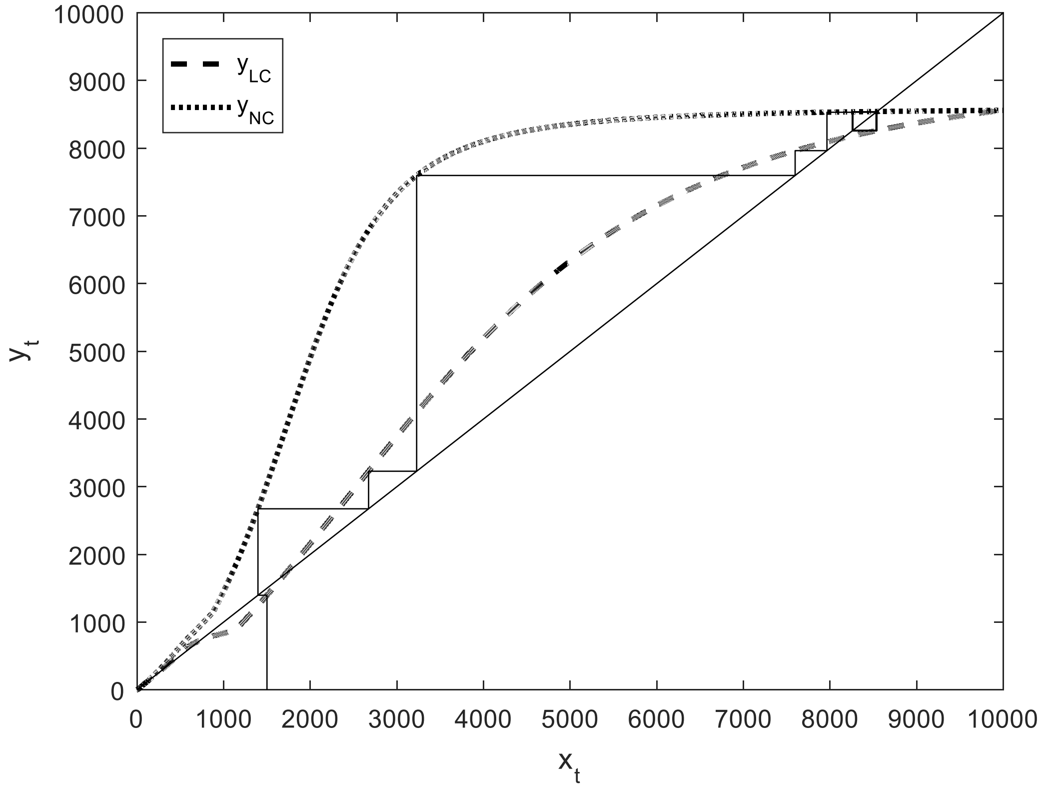

Figure 2: Decision rules for the two intervals (y_{LC}, y_{NC}), the replacement curve (45-degree line), and the dynamic path (solid line) for initial capital level $x_0 = 1{,}500$.

At capital levels from approximately 750 to 1,500, y_{LC} is below the replacement curve. In this region, the exploitation level is higher than the natural growth ($u_k > F_k(x_k)$), and this creates a local attractor near these capital levels. This phenomenon is shown in figure 3, where y_{LC} and y_{NC} together with a dynamic path from the initial capital level $x_0 = 500$ is shown. As seen in the figure, the path is trapped at levels near $x_k = 1{,}000$. Thus, if the capital level is low prior to the implementation of this scheme, it may get trapped at low levels. That is, there is a poverty trap. Dynamic behavior as shown in Figure 3 is not expected for optimized systems like this (Clark 1990) and serves as a simple example of potentially dramatic practical consequences of considering periodic features in optimization problems. The results also suggest that if regulations are based on an approach that abstracts from underlying, periodic features, while rational economic agents optimally adapt to the periodicity, outcomes may be significantly disturbed.

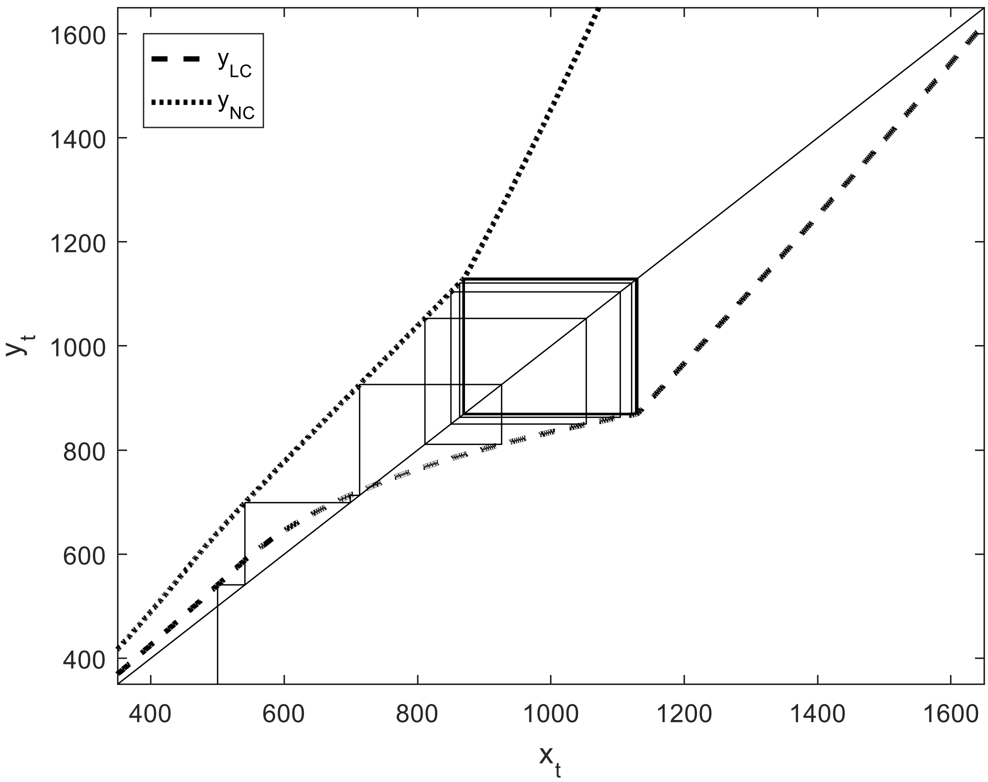

Figure 3: Decision rules at low capital levels for the two intervals (y_{LC}, y_{NC}), the replacement curve (45-degree line), and the dynamic path (solid line) for initial capital level $x_0 = 500$.

5. Final remarks

A method to solve periodic optimization problems is potentially a valuable tool in a wide range of settings. The resource capital example suggests that complex and untypical dynamics arise for a relatively modest deviation from the autonomous formulation. Figure 3 shows that the optimal periodic solution has a trap at low capital levels (that is, a local attractor). Low capital levels in the long run are usually undesirable in most natural capital problems. Moreover, abstracting from periodicity, for example by heuristic approaches like considering an average effect rather than a periodic effect, may quickly lead astray. Further examples show that such heuristics have severe, adverse consequences if decisions are based on an autonomous approximation while agents, subject to these decisions, observe and adapt to the periodic phenomenon. Inter-annual or within-season inefficiencies that agrees well with these examples are observed in empirical studies of fisheries and have gained considerable attention (Smith 2012, Huang and Smith 2014).

Our results are an intuitive extension of the Bellman result. The classical Bellman result is valid for a scalar value function. The periodic problems given in (3) and implied by (5), (6), and (7) are non-autonomous, their value functions are autonomous vector functions, and the Bellman result does not apply. But when the periodic cycle is perceived as the time unit, periodic problems can be perceived as autonomous in higher dimension.

Acknowledgements

We acknowledge grant no. 021-ABEL-CM-2013 from EEA Financial Mechanism (Iceland, Lichtenstein, and Norway), and grant number 257630 from the Research Council of Norway. Part of the computations for this research was performed in EOLO, the High Performance Cluster of Climate Change of the International Campus of Excellence of Moncloa, Universidad Complutense de Madrid, which is funded by MECD and MICINN.

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Consider an infinite horizon, multi-dimensional optimization problem with arbitrary but finite periodicity in discrete time. The problem can be posed as a set of coupled equations. We show that the problem is a special case of a more general class of problems, that the general class has a unique solution, and that the solution can be obtained with the help of a contraction operator. Special cases include the classical Bellman problem and stochastic problem formulations. Thus, we view our approach as an extension of the Bellman problem to the special case of non-autonomy that periodicity represents, and we thereby pave the way for consistent and rigorous treatment of, for example, seasonality in discrete, dynamic optimization. We demonstrate our method in a simple example with periodic variation in the objective function.

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